Turbulent portion of the boundary layer

In turbulent boundary layer neglect the existence of buffer layer. The region near to the surface is laminar sub-layer this region is assumed to be neglected. So that the motion is effected entering by eddy motion. Assumed that the shear stress at a plane surface can be calculated from the expression developed by Blasius:

$$\frac{R}{\rho\, u_s^2} = 0.228\left(\frac{\mu}{\rho\,\delta u_s}\right)^{0.25} \qquad (11)$$

δ: thickness of boundary layer developed by Blasius using power law $\quad \frac{u_x}{u_s} = \left(\frac{y}{\delta}\right)^f \qquad (12)$

By substitute for value of the $u_s = u_x\left(\frac{y}{\delta}\right)^f$ in equation (12) into equation (11):

$$R = 0.0228\ \rho^{0.75}\mu^{0.25}\delta^{-0.25}\quad u_x^{1.75}\left(\frac{\delta}{y}\right)^{1.75f}$$

By using dimension analysis [0= - 0.25+1.75 f) the $f = \frac{1}{7}$

$$\frac{u_x}{u_s} = \left(\frac{y}{\delta}\right)^{\frac{1}{7}} \qquad (13)$$

Assumptions**:** known as Prandtl seventh law which is used to calculate the turbulent boundary layer

Through using equation 13 and combined with equation (2) we can get:

$$\frac{\partial}{\partial x}\int_0^l \rho\,(u_s - u_x)\,u_x dy \;=-R_o \qquad (2)$$

$$u_x = u_s\left(\frac{y}{\delta}\right)^{\frac{1}{7}} \qquad \mathbf{(13)}$$

$$\int_0^l (u_s - u_x)\,u_x dy \;= \int_0^\delta u_s^2\left(1-\left(\frac{y}{\delta}\right)^{\frac{1}{7}}\right)\left(\frac{y}{\delta}\right)^{\frac{1}{7}} dy + \int_\delta^l \cancel{(u_s - u_s)}.\,u_s\,dy$$

$$= u_s^2\int_0^\delta \left(\frac{y}{\delta}\right)^{\frac{1}{7}} - \left(\frac{y}{\delta}\right)^{\frac{2}{7}}\,dy \;= u_s^2\delta\left(\frac{7}{8}-\frac{7}{9}\right)$$

$$\int_0^l (u_s - u_x)\,u_x dy = \frac{7}{72}u_s^2\,\delta \qquad 14$$

Substitute equation (14) and equation (11) in equation (2):

$$\rho\frac{\partial}{\partial x}\left(\frac{7}{72}u_s^2\,\delta\right) = 0.228\rho\,u_s^2\left(\frac{\mu}{\rho\,\delta u_s}\right)^{0.25}$$

$$\boldsymbol{\frac{\delta}{x} = 0.376 Re_x^{-0.2}} \qquad \mathbf{14} \qquad \text{For turbulent local thickness}$$

δ:include laminar and buffer layer(laminar sub-layer is vary thin) so the velocity is constant .

Laminar sub- layer

Assume at x=x the laminar sub-layer is of thickness δ_b and the total thickness of the bounder layer is (δ) . Since the laminar sub-layer is very thin so the velocity gradient and shear stress are ***constant*** inside this layer

$$R_o = -\mu(\frac{\partial u_x}{\partial u})_{y=0} = -\mu\frac{u_x}{y} \qquad 15$$

Where y = δ_b

By substitute equation 15 into equation 11

$$\mu\frac{u_x}{y} = 0.0228\rho\, u_s^2(\frac{\mu}{\rho\,\delta u_s})^{0.25}$$

$$u_x = 0.0228\rho\, u_s^2\ \frac{y}{\mu}\ (\frac{\mu}{\rho\,\delta u_s})^{0.25}$$

At the leading edge $u_x=u_b$ and $y=\delta_b$

$$u_b = 0.0228\rho\, u_s^2\ \frac{\delta_b}{\mu}\ \left(\frac{\mu}{\rho\,\delta u_s}\right)^{0.25}$$

$$= 0.0228\, \rho\, u_s^2\ \frac{\delta_b}{\mu}\left(\frac{\mu}{\rho\,\delta u_s}\right)^{1}\left(\frac{\mu}{\rho\,\delta u_s}\right)^{-3/4}$$

$$\frac{\delta_b}{\delta} = \frac{1}{0.0228}\left(\frac{u_b}{u_s}\right)\left(\frac{\mu}{\rho u_s \delta}\right)^{-\frac{3}{4}} \qquad \mathbf{16}$$

This equation represents the thickness ratio of sub-layer to turbulent boundary layer.

At the leading edge of the laminar sub-layer:

from equation (13) $\frac{u_b}{u_s} = \left(\frac{\delta_b}{\delta}\right)^{\frac{1}{7}}$ (13b) with re-arrange, then substitute in equation (16)

$$\left(\frac{u_b}{u_s}\right)^7 = \frac{1}{0.0228}\left(\frac{u_b}{u_s}\right)\left(\frac{\mu}{\rho u_s \delta}\right)^{\frac{-3}{4}}$$ ($X^7 / X = A\ y^{-3/4}$)

$$\frac{u_b}{u_s} = 1.87\ Re_\delta^{-1/8} \qquad (17)$$

Where $Re_\delta = \frac{u_s \rho \delta}{\mu}$ *and δ is the thickness of turbulent layer*

The thickness in fully turbulent layer is given by equation $\frac{\delta}{x} = 0.376 Re_x^{-0.2}$ *therefore from this equation :*

$$\delta = 0.376\ x^{0.8}\left(\frac{\mu}{\rho u_s}\right)^{0.2} \qquad (18)$$

Substitute the equation (18) in equation (17) to get:

$$\frac{u_b}{u_s} = 1.87\left[\frac{u_s \rho}{\mu}\ 0.376\ \frac{x^{0.8}\mu^{0.2}}{u_s^{0.2}\rho^{0.2}}\right]^{-1/8} \qquad (19)$$

$$\frac{u_b}{u_s} = 2.11\ Re_x^{-0.1} \qquad (20)$$

Notes

$Re_x = \frac{u_s \rho x}{\mu}$, *where* x *is the distance of flow,* u_s *is the mainstream velocity(constant),* u_b *is the velocity inside the laminar sub-layer.*

If the thickness is given in question, then use the equation (17), but if the distance is given (20) is used.

From equations: (13b, 18 , and 20) for δ *:*

$$\frac{\delta_b}{\delta} = \left(\frac{u_b}{u_s}\right)^7 = \frac{190}{Re_x^{0.7}}$$

$$\frac{\delta_b}{x} = \frac{190}{Re_x^{0.7}} \cdot \frac{0.376}{Re_x^{0.2}}$$

$$\boxed{\frac{\delta_b}{x} = 71.5\ Re_x^{-0.9} \qquad (21) \qquad \text{for thickness calculating}}$$

Shear stress at the surface, at a distance (***x***) from the leading edge is $= \mu \frac{\boldsymbol{u_b}}{\boldsymbol{\delta_b}}$, - R = R_b

Substitute for u_b equation (20) and $\boldsymbol{\delta_b}$ equation (21)

$\frac{R}{\rho\, u_s^2} = 0.0296\, Re_x^{-0.2}$ (22) for local friction factor

The mean value of $\frac{R}{\rho\, u_s^2}$ over the range of x = 0 to x = x substitute in equation (22)

$$\left(\frac{R}{\rho\, u_s^2}\right)_{\text{mean}} \times x = \int_0^x \left(\frac{R}{\rho\, u_s^2}\right)\, dx = \int_0^x 0.0296\, Re_x\; .dx$$

$$\boldsymbol{\left(\frac{R}{\rho\, u_s^2}\right)_{\text{mean}} = 0.037\, Re_x^{-0.2}} \qquad (23)$$

Mean friction force in laminar sub-layer for x = 0 to x = x

The total shear force acting on the surface is found by adding the forces acting in the streamline ($x < x_c$) [where $\boldsymbol{x_c}$: is the critical distance between the streamline and turbulent regions] and turbulent region ($x > x_c$), this can be done by providing the critical value of Re_x.

Where:

In streamline, $\frac{R}{\rho\, u_s^2} = 0.646\, Re_x^{-0.5}$

And for turbulent $\left(\frac{R}{\rho\, u_s^2}\right)_{mean} = 0.037\, Re_x^{-0.2}$

- In calculating the mean value of $\left(\frac{R}{\rho\, u_s^2}\right)_{mean}$ in turbulent region it was assume that the turbulent layer extended to leading edge. A more accurate value of mean value of $\left(\frac{R}{\rho\, u_s^2}\right)_{mean}$ over the whole surface can be obtained by using the expression for laminar condition from $x = 0$ to $x = x_c$ and for turbulent from $x = x_c$ to $x = x$ therefore the mean value:

- $\left(\frac{R}{\rho\, u_s^2}\right)_{mean} = \frac{1}{x}\left[0.646\, Re_x^{-0.5} . x_c + 0.037\, Re_x^{-0.2}\ x - 0.037\, Re_x^{-0.2} .\ x_c\right]$

- $= 0.037\, Re_x^{-0.2} + Re_x^{-1}\left[0.646\, Re_{x_c}^{0.5} - 0.037\, Re_{x_c}^{0.8}\right]$ (24)

Note

The change from streamline to turbulent conditions in the boundary layer occurs at a certain critical distance from the leading edge . This distance depends on the shape of leading edge .For a given surface the transition takes place at critical value of Reynold number, the critical Re_{x_c} at $(x_c) = 10^5$ at the surface .

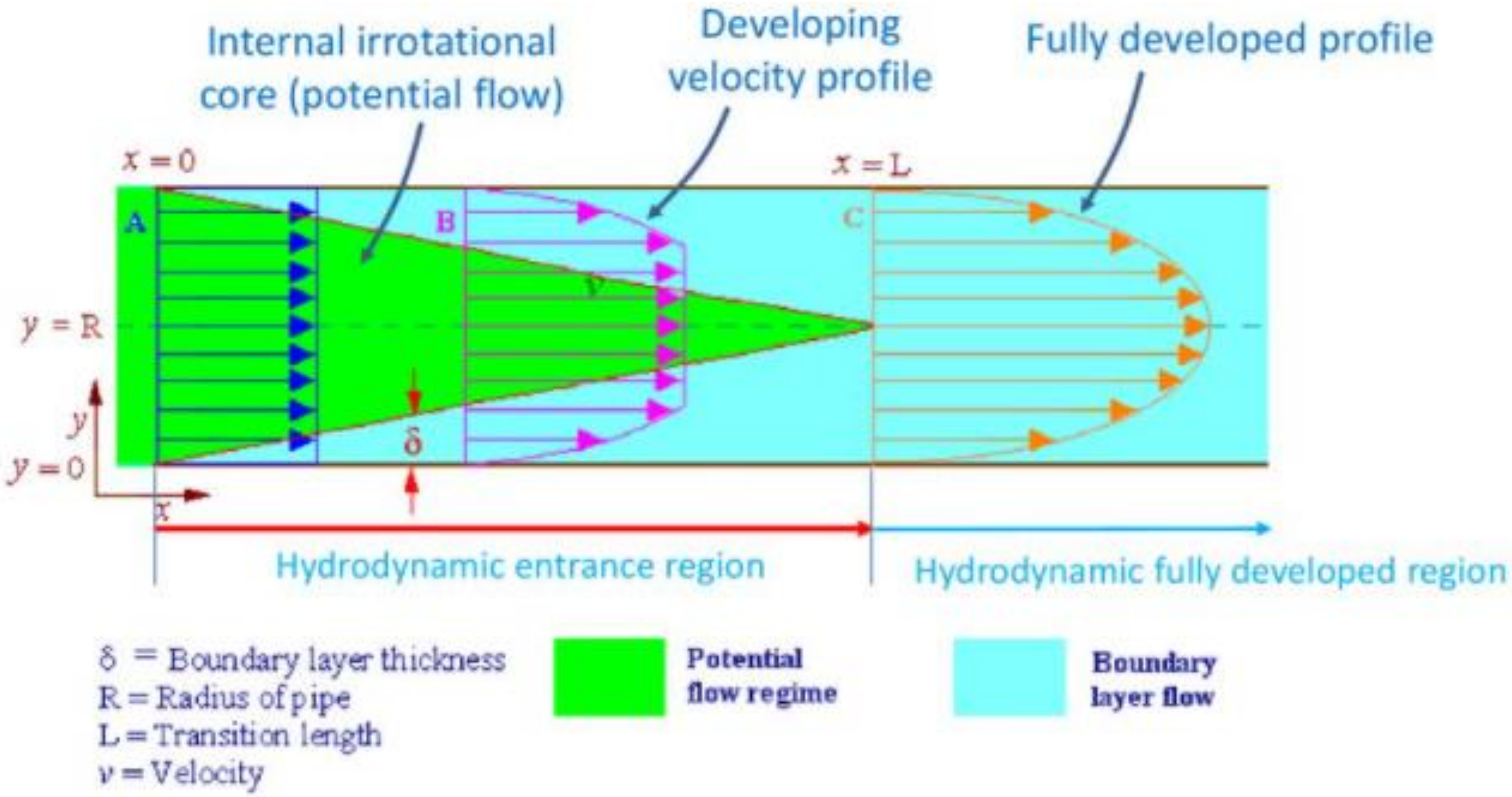
Application of boundary layer theory through a pipe flow :-

Consider a fluid entering a circular pipe at a uniform velocity (fig. in next. slide). Because of the no-slip condition, the fluid particles in the layer in contact with the surface of the pipe come to a complete stop. This layer also causes the fluid particles in the adjacent layers to slow down gradually as a result of friction. The region of the flow in which the effects of the viscous shearing forces caused by fluid viscosity are occurred, is called as the velocity boundary layer or just the boundary layer.

$u_x = u_s \, (\frac{y}{r})^{\frac{1}{7}}$ where u_s= velocity at the center, maximum velocity.

Shear stress at the walls is given by the Blasius equation.

$\frac{R}{\rho u_s^2}$ =0.0228 $(\frac{\mu}{\rho u_s r})^{\frac{1}{4}}$ substitute u = 0.82u_s , d = 2r



Development of boundary-layer flow in pipe

$$\frac{R}{\rho u^2} = 0.0384 Re^{-\frac{1}{4}}$$ (25) for turbulent $Re = \frac{u\rho d}{\mu}$

The velocity at the edge of laminar sub-layer is $\frac{u_b}{u_s} = 1.87(\frac{\mu}{\rho u_s r})^{\frac{1}{8}}$

$$\frac{u_b}{u_s} = 2.49\ Re^{-\frac{1}{8}}$$ (26) Re=$\frac{u\rho d}{\mu}$

The thickness of laminar sub-layer:

$\frac{\delta_b}{r}$=$(\frac{u_b}{u_s})^{7}$= $(1.78)^{7}$ $(\frac{\mu}{\rho u_s r})^{\frac{7}{8}}$

$$\frac{\delta_b}{d} = 62\ Re^{-7/8}$$ (29)

Summary

- Laminar boundary layer:
- $\frac{\delta}{x} = 4.64 \quad Re_x^{-1/2}$ (8) {velocity profile is polynomial series}. To find the thickness
- $\frac{R_o}{\rho u_s^2} = -0.323\ Re^{-1/2}$ (9) To find shear stress

Turbulent boundary layer

❖ $\frac{\delta}{x} = 0.376 Re_x^{-0.2}$ **14** For turbulent fully developed flow (neglect laminar sub-layer)

❖ $\frac{R}{\rho\, u_s^2} = 0.228 Re_\delta^{-0.25}$

Laminar sub-layer:

To find velocity in presence laminar sublayer:

- ❑ $\frac{\boldsymbol{u_b}}{\boldsymbol{u_s}} = 1.87\ Re_{\delta}^{-1/8}$ (17)
- ❑ Or $\frac{\mathrm{u_b}}{\mathrm{u_s}} = 2.11\ \mathrm{Re_x^{-\,0.1}}$ (20)
- ❑ $\frac{\boldsymbol{\delta_b}}{\boldsymbol{x}} = \mathbf{71.5}\ \boldsymbol{Re_x^{-\,0.9}}$ (21) for thickness calculating
- ❑ $\frac{R}{\rho\, u_s^2} = 0.0296\ Re_x^{-0.2}$ (22) for local friction factor
- ❑ $\left(\frac{\boldsymbol{R}}{\boldsymbol{\rho\, u_s^2}}\right)_{\textbf{mean}} = 0.037\ Re_x^{-0.2}$ (23)

HW

Calculate the thickness of the boundary layer at a distance of 75 mm from the leading edge of a plane surface over which water is flowing at a rate of 3 m/s. Assume that the flow in the boundary layer is streamline and that the velocity u of the fluid at a distance y from the surface can be represented by the relation $u = a + by + cy^2 + dy^3$, where the coefficients a, b, c, and d are independent of y. The viscosity of water is 1 mN s/m^2.

Ans. $\boldsymbol{\delta}$ = 0.000734 m or 0.734 mm